Chapter 5

Generic Second Order System

Second order systems can be mathematically tractable to a great depth and derive mathematical expressions for system response. Using those expressions a feedback controller can be designed for second order systems very effectively. Most industrial plants can be accurately modeled as second order systems. The generic second order model is shown in Fig.5

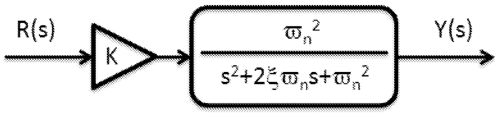

Figure 5.1: Generic second order model

The generic second order system is

$$G_{g2}(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{5.1}$$

where ζ and ω_n are *damping ration* and *natural undamped frequency*. In the height controlled stabilized table discussed earlier, the closed loop transfer function $\frac{\eta}{s^2+2\sigma s+\rho+K\eta}$ is in fact presented as a generic second order system $\frac{1}{k+K} \cdot \frac{(k+K)/m}{s^2+(b/m)s+(k+K)/m}$ after substitution for $\sigma = b/2m, \rho = k/m$, and $\eta = 1/m$. The three system parameters m,k,b determine the two generic second order parameters $\omega_n = \sqrt{\frac{k+K}{m}}$, and $\zeta = \frac{b}{2\sqrt{m(k+K)}}$. The DCG of the generic second order system $\lim_{s\to 0} s \frac{(k+K)/m}{s^2+(b/m)s+(k+K)/m} = 1$, whereas the actual system has $DCG = \frac{1}{k+K}$. The characteristic equation of the generic second order system is

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0 \tag{5.2}$$

and the poles are

$$s_1, s_2 = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \tag{5.3}$$

The response then is determined by ζ as follows.

1. ζ >1 over damped (slow, no oscillations)
2. ζ =1 critically damped (quickest, nonoscillatory)
3. 0< ζ <1 under damped (damped oscillations)
4. ζ =0 stable sustained oscillations (simple harmonic motion)
5. ζ <0 unstable response

out of all these possible responses, the damped oscillations $0 < \zeta < 1$ is the most interesting response for mathematical analysis. This response is very common in industrial process control plants, where quick response with some oscillations is acceptable. The next section will thoroughly analyze this response.

5.1 Under-damped Generic Second Order System ($0 < \zeta < 1$)

In this response, the two poles in (5.3) can be written as follows.

$$\begin{aligned} s_1, s_2 &= -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} \\ &= -\zeta\omega_n \pm j\omega_d \end{aligned} \tag{5.4}$$

where $\omega_d = \omega_n\sqrt{1-\zeta^2}$ is the *damped frequency.* The two poles can be illustrated as in Fig.5.2. With the complex conjugate pair of poles, generic second order system (5.1) can be written as follows.

$$\begin{aligned} G_2(s) &= \frac{\omega_n^2}{(s+\zeta\omega_n+j\omega_d)(s+\zeta\omega_n-j\omega_d)} \\ &= \frac{\omega_n^2}{(s+\zeta\omega_n)^2+\omega_d^2} \\ &= \frac{\omega_n^2}{\omega_d}\frac{\omega_d}{(s+\zeta\omega_n)^2+\omega_d^2} \end{aligned} \tag{5.5}$$

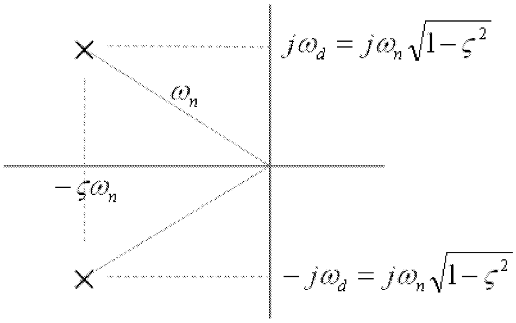

Figure 5.2: Pole pair of a under-damped generic second order system

The unit step $(\frac{1}{s})$ response of the second order system is therefore

$$Y(s) = \frac{\omega_n^2}{\omega_d}\frac{\omega_d}{(s+\zeta\omega_n)^2+\omega_d^2}.\frac{1}{s} \tag{5.6}$$

The response in time-domain can be obtained by inverse Laplace transformation of (5.6). The middle term of (5.6) looks similar to the Laplace transform of $sin(\omega_d t)$ except that s being replaced by $s+\zeta\omega_n$, which is the result of an exponential scaling $e^{-\zeta\omega_n t}$ in time-domain. Thus, the inverse Laplace transform of the middle term should be $e^{-\zeta\omega_n t}\sin(\omega_d t)$. However, the presence of $1/s$ term confirms that it should be the integral of the inverse Laplace transform of the second term. Therefore, inverse Laplace transform of (5.6) should be $\int e^{-\zeta\omega_n t}\sin(\omega_d t)dt$. This integral has been evaluated in (A.3) in Appendix.A for $\zeta\omega_n = \sigma$ and $\omega_d = \omega$. Using (A.3) and substituting for $\sigma = \zeta\omega_n$, and $\omega = \omega_d$, the response can be determined as follows

$$y(t) = \frac{\omega_n^2}{\omega_d}\left[\frac{\omega_d}{\omega_d^2+\zeta^2\omega_n^2} - \frac{e^{-\zeta\omega_n t}}{\sqrt{\omega_d^2+\zeta^2\omega_n^2}}\sin(\omega_d t+\phi)\right] \tag{5.7}$$

where $\tan\phi = \frac{\omega_d}{\zeta\omega_n} = \frac{\sqrt{1-\zeta^2}}{\zeta}$. As $\omega_d = \omega_n\sqrt{1-\zeta^2}$. Also because $\zeta^2\omega_n^2+\omega_d^2 = \zeta^2\omega_n^2+\omega_n^2(1-\zeta^2) = \omega_n^2$,

$$\begin{aligned} y(t) &= \frac{\omega_n^2}{\omega_d}\left[\frac{\omega_d}{\omega_n^2} - \frac{e^{-\zeta\omega_n t}}{\omega_n}\sin(\omega_d t+\phi)\right] \\ &= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t+\phi) \end{aligned} \tag{5.8}$$

This is the unit step response of an under damped $(0 < \zeta < 1)$ generic second order system.

Peak Time t_p, and Rise Time t_r

Peak time t_p is the time that the response takes to reach its first peak. The rise time t_r is the time between 10% and 90% of the response in the first peak. By differentiation of (5.8),

$$\begin{aligned}
\dot{y}(t) &= -\frac{1}{\sqrt{1-\zeta^2}}\left[\omega_d e^{-\zeta\omega_n t}\cos(\omega_d t+\phi) - \sin(\omega_d t+\phi)e^{-\zeta\omega_n t}\zeta\omega_n\right] \\
&= \frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\left[\zeta\sin(\omega_d t+\phi) - \sqrt{1-\zeta^2}\cos(\omega_d t+\phi)\right] \\
&= \frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\left(\omega_d t+\phi-\phi\right) \\
&= \frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\left(\omega_d t\right) \qquad (5.9)
\end{aligned}$$

when $t = t_p$ $\dot{y(t)}$=0, therefore sin($\omega_d t_p$)=0, which yields to

$$\begin{aligned}
\omega_d t_p &= \pi \\
\omega_n\sqrt{1-\zeta^2}t_p &= \pi \\
t_p &= \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} \qquad (5.10)
\end{aligned}$$

A rule of thumb for rise time can be determined if we set a reasonable value for ζ. It is also reasonable to assume that $t_r \approx \frac{t_p}{2}$. Therefore, by setting $\zeta = 0.5$, the rule of thumb for rise time is

$$t_r \approx \frac{1.8}{\omega_n} \qquad (5.11)$$

Peak Overshoot PO

Peak value of $y(t)$ in (5.8) occurs at $t = t_p$

$$1 - \frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t_p}\sin\left(\omega_d t_p+\phi\right) = 1 + PO \qquad (5.12)$$

where PO is the percentage overshoot. From (5.10) $\omega_d t_p = \pi$, and therefore $\omega_n t_p = \frac{\pi}{\sqrt{1-\zeta^2}}$. With these two equivalent expressions substituted in (5.12) we have

$$-\frac{1}{\sqrt{1-\zeta^2}}e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}}\sin\left(\pi+\phi\right) = PO \qquad (5.13)$$

Because $\sin \pi + \phi = -\sin \phi = \sqrt{1-\zeta^2}$, (5.13) can be written as follows.

$$\begin{aligned}
\frac{1}{\sqrt{1-\zeta^2}} e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} \sqrt{1-\zeta^2} &= PO \\
e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} &= PO \\
-\frac{\zeta\pi}{\sqrt{1-\zeta^2}} &= \ln PO \qquad (5.14) \\
\tan\beta &= -\frac{\ln PO}{\pi} \\
\beta &= \tan^{-1}\left\{-\frac{\ln PO}{\pi}\right\} \qquad (5.15)
\end{aligned}$$

where damping angle $\beta = \pi/2 - \phi$

Settling Time t_s

Settling time t_s is the time required for the oscillation to decay down to 1% of the steady state level. Therefore, referring to (5.8)

$$\begin{aligned}
e^{-\zeta\omega_n t_s} &\approx 0.01 \\
-\zeta\omega_n t_s &\approx \ln 0.01 \\
t_s &\approx \frac{4.6}{\zeta\omega_n} \qquad (5.16)
\end{aligned}$$

5.2 Desired Response

In system design, the maximum overshoot PO_{max} is specified, and from (5.15) the corresponding damping angle can be determined. The actual design needs to make sure that the oscillations are controlled below this level, and therefore the damping angle should obey the following constraint.

$$\beta > \tan^{-1}\left\{-\frac{\ln PO_{max}}{\pi}\right\} \qquad (5.17)$$

Furthermore, if a maximum settling time $t_{s,max}$ is specified, the corresponding decay constant $\zeta\omega_n$ can be determined from (5.16). The actual system has to settle before this time, therefore the constraint for decay constant can be determined as follows.

$$\zeta\omega_n > \frac{4.6}{t_{s,max}} \qquad (5.18)$$

Finally, if the response has to rise before a maximum time of $t_{r,max}$, the oscillation can be improved according to (5.11) as follows.

$$\omega_n > \frac{1.8}{t_{r,max}} \tag{5.19}$$

Considering these three constraints, the appropriate region to locate the two poles can be determined as illustrated as in Fig.5.2.

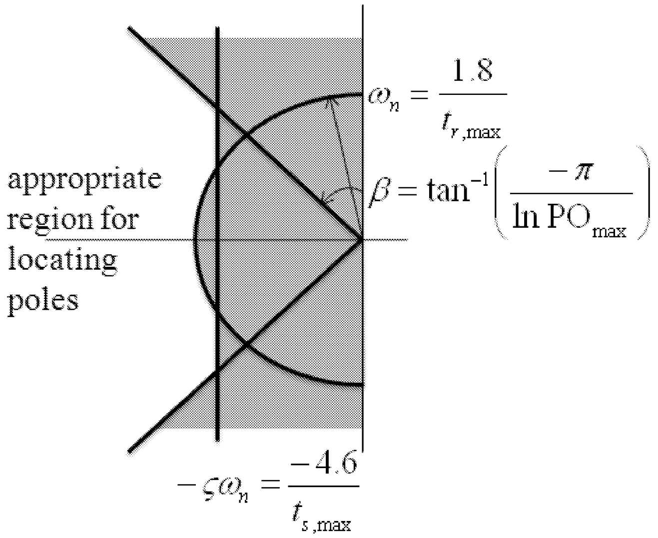

Figure 5.3: ζ and ω constraints for desired response

5.3 Example

A generic second order system is given by $\frac{43}{s^2+3s+43}$ in that $\omega_n = \sqrt{43} =$ 6.56[rad/s], $\zeta = \frac{3}{2\omega_n}$ =0.23, and $\phi = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta} = 1.34$[rad]. From (5.8), the response can be determined as $y(t) = 1 - 1.03e^{-1.5t}\sin(6.38t + 1.34)$. Response rise time $t_r \approx \frac{1.8}{\omega_n} = 0.27$[s], and percentage overshoot $PO = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} = 0.48$, and settling time $t_s \approx \frac{4.6}{\zeta\omega_n} = 3.01$[s] can be calculated. The response is illustrated in Fig.5.4.

The response is not desirable due to the presence of significant overshoot and long settling time. If the overshoot has to be controlled below 5% (i.e. PO_{max} =0.05), and response has to settle before 2s (i.e. $t_{s,max}$=2). Then, from (5.18), $\zeta\omega_n > \frac{4.6}{2}$. Lets select $\zeta\omega_n$=2.5, and from

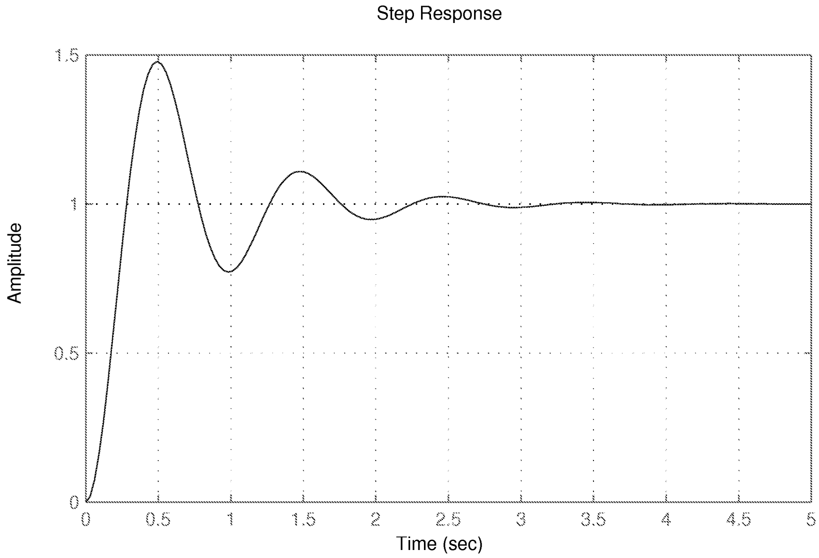

Figure 5.4: Second order generic response

(5.17) $\beta > \tan^{-1}\{-\ln 0.05/\pi\} = 46.36^o$. Lets select $\beta = 50^o$. Because $\omega_n\sqrt{1-\zeta^2} = \frac{\zeta\omega_n}{\tan\phi} = \frac{2.5}{\tan 50^o} = 2.1$, the two poles can be determined as $-\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} = -2.5 \pm j2.1$. The desired characteristic equation is $s^2+2\times 2.5s+(2.5^2+2.1^2) = s^2+5s+10.66$, and desired generic second order system is $\frac{10.66}{s^2+5s+10.66}$. Then, the resulting settling time is t_s=4.6/2.5=1.84[s], and peak overshoot from (5.14) $PO = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} = e^{-\frac{2.5\pi}{2.1}} = 0.02$, and from (5.10) peak time t_p=1.26[s], which are all acceptable. The desired response is illustrated in Fig.5.5

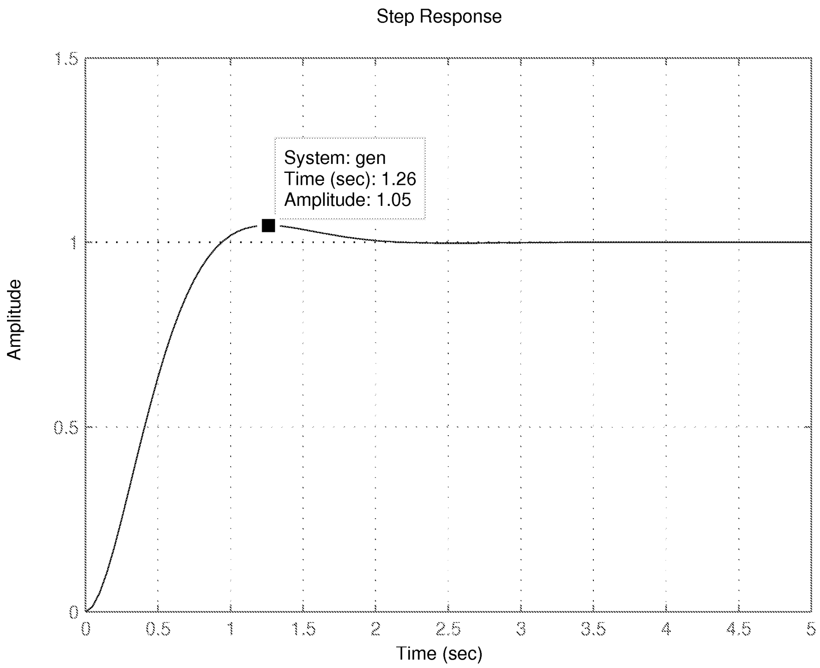

Figure 5.5: Second order desired response